manipulation; whilst the extent of my obligation to Professor Turner is very far indeed from being covered by the definite acknowledgments I have already made.

Some Suggestions for the Explicit Use of Direction Cosines or Rectangular Coordinates in Astronomical Computations. By H. H. Turner, M.A., F.R.S., Savilian Professor.

1. Until the introduction of the photographic method, the simple computations of astronomy could be fairly regarded as settled in form. The form depended essentially on the large use of meridian observations, which determine R.A. and N.P.D. independently, and thus all computations and catalogues are arranged in terms of R.A. and N.P.D. Tables of the planets, which are most conveniently expressed in latitude and longitude, and are so expressed in the first instance, are converted into terms of R.A. and N.P.D. for the use of observers.

2. But even independently of photography some inconveniences attending the use of these particular coordinates have made themselves felt. The neighbourhood of the pole is always a difficulty, and calls for some modification of the ordinary processes. Fabritius proposed in *Ast. Nach.* Nos. 2072–3 to use the coordinates

$$x = \sin p \cos\alpha \qquad y = \sin p \sin\alpha$$

in the neighbourhood of the pole, where α, p are the R.A. and N.P.D. of a star; and he showed that the use of these rectangular coordinates considerably simplifies the calculation of precession and nutation for polar stars. His x and y may be thus regarded. Draw a tangent plane to the sphere at the pole, and orthogonally project stars from the sphere on to this plane by lines parallel to the Earth's axis; then (x, y) are the Cartesian coordinates of the projection referred to axes in this plane.

3. In connection with the measurement and reduction of photographic plates I have urged the advantages of using "standard coordinates" which are also the Cartesian coordinates of a star supposed projected on a tangent plane to the celestial sphere; but in this case the projection is not orthogonal; it is by lines radiating from the centre of the sphere. For a plate with centre at the pole the "standard coordinates" would be

$$\xi = \tan p \cos\alpha \qquad \eta = \tan p \sin\alpha$$

as compared with Fabritius's

$$x = \sin p \cos\alpha \qquad y = \sin p \sin\alpha$$

4. Other projections on the tangent plane might be made. Thus the well-known stereographic projection would be represented by

$$X = 2 \tan\frac{p}{2} \cos\alpha \qquad Y = 2 \tan\frac{p}{2} \sin\alpha$$

and it is possible that these coordinates or yet others may be found useful in some department of astronomical work. But without following generalities further in this direction, we may notice that both Fabritius's and standard coordinates are closely related to direction cosines. If the three direction cosines of a star be

$$l = \sin p \cos a \qquad m = \sin p \sin a \qquad n = \cos p$$

then we have for Fabritius coordinates

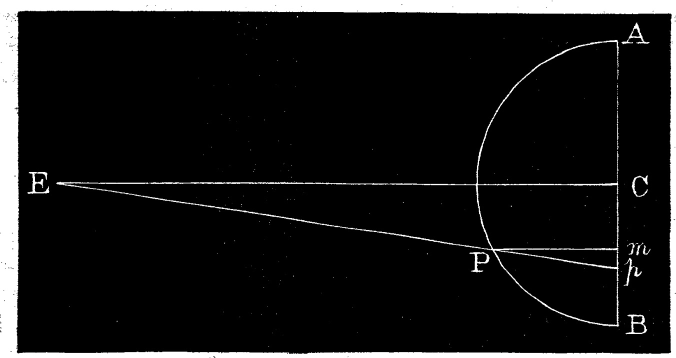
$$x = l \qquad y = m$$

and for standard coordinates

$$\xi = \frac{l}{n} \qquad \eta = \frac{m}{n}$$

Now this suggests that the more explicit use of direction cosines in our computations may be attended with advantages; and the present paper is written with the object of calling attention to this point by two particular examples.

5. The first example is that in which we are dealing with



positions of objects on the surface of the Sun or Moon. Let C be the centre of the Moon; E, the spectator on the Earth. He sees any point P on the surface of the Moon projected on the plane AB, as at p. Now E is at a great distance, and hence EP is *nearly* parallel, though not quite, to EC. By a slight correction to Cp we can find the orthogonally projected coordinates of P, and are thus prepared for the use of direction cosines in the reductions. The correction required is

$$\begin{aligned} pm &= \mathrm{P}m \tan m\mathrm{P}p \\ &= \sqrt{\mathrm{CB}^2 - \mathrm{C}m^2} \,.\, \mathrm{C}p \,.\, /\mathrm{CE} \\ &= \frac{\mathrm{CB}}{\mathrm{CE}} \left\{ 1 - \left(\frac{\mathrm{C}m}{\mathrm{CB}}\right)^2 \right\}^{\frac{1}{2}} \mathrm{C}p \end{aligned}$$

Here CB/CE is the angular radius of the Moon in circular measure—about ·005. The remainder of the expression has a maximum value when $\mathrm{C}m^2 = \frac{1}{2}\mathrm{CB}^2$, nearly; and thus the maximum value of the correction is

$$\tfrac{1}{2} \times {\cdot}005 \times 16' = 2''{\cdot}5, \text{ say.}$$

Hence a small table suffices to give the fraction by which we must diminish Cp.

If, then, we measure on the Moon, or on a photograph of the Moon, rectangular coordinates x' and y' from the centre of the disc, and expressed in terms of the Moon's radius, and with $\sqrt{x'^2+y'^2}$ as argument, take out from the table the appropriate reducing factor, we can get x, y, two of the direction cosines of the point, and can then find $z = \sqrt{1-x^2-y^2}$.

6. These (x, y, z) are coordinates of the point with axis of z directed to the centre of the Earth and axes of x and y at right angles to it, with some orientation depending upon the way in which the measures were made. Let now (X, Y, Z) be direction cosines of the point referred to chosen axes fixed in the Moon. Then by the properties of direction cosines we know that

$$\xi = a_1x + a_2y + a_3z$$
$$\eta = b_1x + b_2y + b_3z$$
$$\zeta = c_1x + c_2y + c_3z$$

with the reciprocal relations

$$x = a_1\xi + b_1\eta + c_1\zeta$$
$$y = a_2\xi + b_2\eta + c_2\zeta$$
$$z = a_3\xi + b_3\eta + b_3\zeta$$

where $(a_1, a_2, a_3 \ldots c_3)$ are a system of direction cosines expressing the relationship of the two systems. We may calculate these coefficients from our knowledge of the Moon's librations; or, if we know the coordinates (X, Y, Z) of certain points on the Moon, of which we determine the (x, y, z) by measures, we can find the coefficients $(a_1 \ldots c_3)$ by substituting for these known points in the equations above given.

7. Again, our measures may only give us x and y, affected with certain errors—those due to refraction, or defective centring of the plate, or defective scale value, or orientation. We shall thus get, not x and y, but two linear functions of them and z, viz.

$$x_0 = \mathrm{A}x + \mathrm{B}y + \mathrm{C}z + \mathrm{D}$$
$$y_0 = \mathrm{E}x + \mathrm{F}y + \mathrm{G}z + \mathrm{H}$$

where the coefficients AB ... H are not altogether known, though certain parts of them, such as that due to refraction, may be calculated. Without troubling to make this calculation, however, we see from the linearity of all the above relations that we may write

$$x_0 = a\xi + b\eta + c\zeta + d$$
$$y_0 = e\xi + f\eta + g\zeta + h$$

so that the measures x_0 and y_0 (corrected to orthogonal projection of course) are expressible as linear functions of the coordinates (ξ, η, ζ). If, then, we know the (ξ, η, ζ) of a certain number of measured points we can find these eight constants of the plate $(a, b \ldots h)$, and then using them for any other measured point we can find the (ξ, η, ζ) from the two equations of the above form combined with (ξ, η, ζ).

8. Of this method Mr. Saunder (see p. 184) has made trial in measuring some beautiful lunar photographs, kindly lent by the Director of the Paris Observatory (M. Loewy). He found the method unsatisfactory, however, because there are, in fact, at present not enough points on the Moon whose positions are well determined. Mr. Saunder has therefore devised (see p. 185) an elegant method of his own, which bids fair to give us excellent positions of a number of points; and then it is possible that he may return to this method for finding the coordinates of other points.

9. But there is a point of detail in which the method requires improvement, viz. it is rather troublesome to solve the three equations

$$a\xi + b\eta + c\zeta = x_0 - d = x_1, \text{ say}$$
$$e\xi + f\eta + g\zeta = y_0 - h = y_1, \text{ say}$$
$$\xi^2 + \eta^2 + \zeta^2 = 1$$

(of which the third is a quadratic) for every point. The constants d and h are quickly subtracted, of course, and may be disregarded. To avoid the solution of the quadratic I suggest the following procedure :—

But for refraction and accidental errors the constants $a, b \ldots g$ would fulfil the relations

$$a^2 + b^2 + c^2 = 1,\ e^2 + f^2 + g^2 = 1,\ ae + bf + cg = 0$$

and then putting

$$1 - a^2 - e^2 = p^2,\ 1 - b^2 - f^2 = q^2,\ 1 - c^2 - g^2 = r^2,\ 1 - x_1{}^2 - y_1{}^2 = z_1{}^2$$

we should have

$$\xi = ax_1 + ey_1 + pz_1$$
$$\eta = bx_1 + fy_1 + qz_1$$
$$\zeta = cx_1 + gy_1 + rz_1$$

All these calculations could be quickly made, especially with an arithmometer and a table of squares.

Owing to refraction and accidental errors these relations are only approximately fulfilled. But if instead of x_1 and y_1 we write

$$x_1 + \lambda x_1 + \mu y_1 \text{ and } y_1 + \mu x_1 + \nu y_1,$$

where λ, μ, ν are small, we can determine (λ, μ, ν) so as to fulfil the corresponding relations *exactly*. We have, in fact,

$$(a+a\lambda+e\mu)^2+(b+b\lambda+f\mu)^2+(c+c\lambda+g\mu)^2=1$$
$$(e+a\mu+e\nu)^2+\quad . \quad . \quad . \quad . \quad . \quad =1$$
$$(a+a\lambda+e\mu)(e+a\mu+e\nu)+\quad . \quad . \quad . \quad =0$$

Put $\quad 1-(a^2+b^2+c^2)=\text{A},\ 1-(e^2+f^2+g^2)=\text{C},\ ae+bf+cg=-\text{B},$

where A, B, C are at least of the same order as λ, μ, ν; and neglect in the first instance squares of small quantities. The equations become

$$2\lambda=\text{A},\ 2\nu=\text{C},\ 2\mu=\text{B},$$

and the test whether this approximation is sufficient is that the corrected values of the coefficients should sensibly fulfil the above relations. If not, the process may be repeated, and new values of λ, μ, ν found. But since refraction is small, one approximation should suffice. Putting, then,

$$\alpha=a+a\lambda+e\mu \quad \beta=b+b\lambda+f\mu \text{ \&c.}$$

and $\quad x_2=x_1+\lambda x_1+\mu y_1,\ y_2=y_1+\mu x_1+\nu y_1,\ z_2{}^2=1-x_2{}^2-y_2{}^2,$

so that α, β and also x_2, y_2, z_2 can be found without much trouble, we have

$$\xi=\alpha x_2+\epsilon y_2+\pi z_2$$
$$\eta=\beta x_2+\zeta y_2+\kappa z_2$$
$$\zeta=\gamma x_2+\eta y_2+\rho z_2$$

and thus ξ, η, ζ can be found simply.

10. A second example of the explicit use of linear formulæ of this kind is afforded by the computations of the effect of precession and nutation. The advantages have been partially pointed out by Fabritius and others. In *Ast. Nach.*, No. 3610, recently published, Dr. W. Ebert considers the special application of Fabritius's formulæ to stars within 20′ of the pole. But I venture to think that the advantages of the *general* application of such formulæ have been hitherto overlooked. Precession and nutation simply change the three axes of reference, so that if (x, y, z) are the direction cosines of a star for one date and (ξ, η, ζ) those for another, we have

$$\xi=a_1x+a_2y+a_3z$$
$$\eta=b_1x+b_2y+b_3z$$
$$\zeta=c_1x+c_2y+c_3z$$

where the constants a_1, a_2, a_3 . . . c_3 are connected by the six well-known relations. Since the axes of reference change slowly

a_1, b_2, c_3 are nearly unity, and the others nearly zero, for any moderate interval. We may write, in fact,

$$\xi - x = ax + a_2 y + a_3 z$$
$$\eta - y = b_1 x + \beta y + b_3 z$$
$$\zeta - z = c_1 x + c_2 y + \gamma z$$

and then the coefficients on the right are all small, and only approximate values of x, y, z are needed on the right.

11. Some of the coefficients are of the first order and some of the second. To show their general character let us neglect the slow motion of the ecliptic and consider the pole of the equator (axis of z or ζ) revolving round the fixed pole of the ecliptic with angular velocity q. Let x, y, z be the coordinates of a star at epoch $t=0$, and ξ, η, ζ the coordinates at time $t=t$.

The pole of the ecliptic has the same coordinates in both systems, viz.

$$x=0 \qquad y=-\sin\omega \qquad z=\cos\omega$$
$$\xi=0 \qquad \eta=-\sin\omega \qquad \zeta=\cos\omega$$

Thus

$$0=-a_2\sin\omega+a_3\cos\omega$$
$$-\sin\omega=-b_2\sin\omega+b_3\cos\omega$$
$$\cos\omega=-c_2\sin\omega+c_3\cos\omega.$$

Therefore

$$\tan\omega=\frac{a_3}{a_2}=\frac{b_3}{b_2-1}=\frac{c_3-1}{c_2}.$$

Multiply the numerators and denominators by a_3, b_3, c_3 respectively and add. Each ratio is found equal to $\frac{1-c_3}{-b_3}$ and this shows that $b_3=c_2$. The same result is obtained if we multiply by a_2, b_2, c_2. But from multiplying by a_1, b_1, c_1 we get each ratio equal to $\frac{c_1}{b_1}$; whence $c_1=b_1\tan\omega$. It will be found that we can now express all the quantities in terms of c_3. Thus

$$c_2 = (c_3-1)\cot\omega$$
$$c_1^2 = 1-c_2^2-c_3^2$$
$$b_1 = c_1\cot\omega$$
$$b_3 = (c_3-1)\cot\omega$$
$$b^2_2 = 1-b_1^2-b_3^2$$

and so on. To find c_3 we must refer to the spherical triangle formed by E, the pole of the ecliptic, and P_1 P_2, the two poles of the equator at the different epochs. In this triangle

$$c_3=\cos P_1P_2=\cos P_1E\cos P_2E+\sin P_1E\sin P_2E\cos E$$
$$=\cos^2\omega+\sin^2\omega\cos qt.$$

Hence $c_2 = \sin\omega\cos\omega(\cos qt - 1)$

$c_1 = \sin\omega\sin qt$

$b_1 = \cos\omega\sin qt.$

We may express the values by the following scheme :—

	x	y	z
ξ	$\cos qt$	$-\cos\omega\sin qt$	$-\sin\omega\sin qt$
η	$\cos\omega\sin qt$	$\cos^2\omega\cos qt + \sin^2\omega$	$\cos\omega\sin\omega(\cos qt - 1)$
ζ	$\sin\omega\sin qt$	$\cos\omega\sin\omega(\cos qt - 1)$	$\sin^2\omega\cos qt + \cos^2\omega.$

12. Now qt is a small quantity. In a century $qt = 5000''$ approximately, or, say, ·025 in circular measure. Thus we may exhibit the approximate values of powers of qt for different periods as follows :—

	10 yrs.	50 yrs.	100 yrs.	300 yrs.
qt	·0025	·0125	·025	·075
$(qt)^2$	·0000063	·0001563	·000625	·005625
$(qt)^3$	·0000000	·0000020	·0000156	·0004188
$(qt)^4$	...	·0000000	·0000004	·0000316
$(qt)^5$	...	...	·0000000	·0000024

Hence, if we are applying precessions for ten years, we can neglect the third power of qt; for fifty years the fourth power; and, indeed, up to one hundred years we can neglect the fourth power, which, as we shall see, is generally multiplied by a small fraction.

13. For simplicity let us first neglect $(qt)^3$—*i.e.* consider precessions for moderate periods comparable with ten or twenty years. Then putting $qt\ \cos\omega = s$, $\tan\omega = r$; since $\cos\omega = ·91$, s is less than qt, and its powers converge more rapidly. Also powers of $r(=0·4)$ converge quickly. The scheme of transformation now becomes

	x	y	z
ξ	$1-\frac{1}{2}(1+r^2)s^2$	$-s$	$-rs$
η	$+s$	$1-\frac{1}{2}s^2$	$-\frac{1}{2}rs^2$
ζ	$+rs$	$-\frac{1}{2}rs^2$	$1-\frac{1}{2}r^2s^2$

Since s is proportional to the first power of the time, and s^2 to the second, we see that this scheme gives in a compendious form both the precessions and secular variations of x, y, z—viz. the precessions are

$$\text{in } x, \quad -(y+rz)s$$

$$\text{in } y, \quad +x\,.\,s$$

$$\text{in } z, \quad +xr\,.\,s$$

and the secular variations are

$$\text{in } x \quad -\tfrac{1}{2}x(1+r^2)s^2$$
$$\text{in } y \quad -\tfrac{1}{2}(y+rz)s^2$$
$$\text{in } z \quad -\tfrac{1}{2}r(y+rz)s^2$$

14. Similarly, the additional terms of the third and fourth orders are readily written down as follows, omitting a common factor, $\frac{1}{24}s^3(1+r^2)$, for convenience in writing :—

	x	y	z
ξ	$s(1+r^2)$	4	$4r$
η	-4	$+s$	$+sr$
ζ	$-4r$	$+sr$	$+s \, . \, r^2$

The terms of the third order are all less than $\frac{1}{6}(qt)^3$, and only begin to affect the seventh place of decimals after about thirty years.

The largest term of the fourth order is $\frac{1}{24}(qt)^4$, and does not affect the seventh place for more than a century.

15. Since the ecliptic is slowly moving, terms must be added to these coefficients, depending on the motion of the ecliptic ; but these will be of a higher order still ; and enough has been said to show the general character of the coefficients. Returning to the general form for them

$$\xi = a_1x + a_2y + a_3z$$
$$\eta = b_1x + b_2y + b_3z$$
$$\zeta = c_1x + c_2y + c_3z$$

Suppose (x, y, z) refer to any standard epoch—say 1900·0. Then, to find the coordinates for any other epoch, we want the nine coefficients, $a_1, a_2 \ldots c_3$. Now it would not be a difficult matter to tabulate these for every year for three hundred years—say, from A.D. 1700 to 2000 (all we are likely to want at present)—and we should then have the means of bringing up to 1900·0 *accurately* any stellar positions referred to another epoch. There is no special difficulty about polar stars ; the formulæ never become less simple in any part of the celestial sphere ; and the labour of multiplication can be quickly performed with an arithmometer, or in many cases by Crelle's Tables, since the coefficients are small.

16. To reduce from one epoch to another when neither is 1900·0, we must form nine new coefficients ; but these are readily formed when we have those connecting both epochs with 1900·0. Thus let (ξ, η, ζ) refer to one epoch, as above, and (X, Y, Z) to another, and let

$$X = A_1x + A_2y + A_3z, \text{ \&c.}$$

so that
$$x = A_1X + B_1Y + C_1Z, \text{ \&c.}$$

according to the usual inversion of direction cosines.

Then $\xi = a_1 x + a_2 y + a_3 z$

$$= (a_1 A_1 + a_2 A_2 + a_3 A_3) X + (a_1 B_1 + a_2 B_2 + a_3 B_3) Y + (a_1 C_1 + a_2 C_2 + a_3 C_3) Z$$

with similar equations for η and ζ, and the law of formation of the coefficients is obvious.

17. *Aberration.*—The use of direction cosines does not always simplify the formulæ, however, and we may take aberration as an instance where there is no great simplification. The effect of aberration is well known to be precisely similar to that of parallax, viz. a displacement of the observer in a given direction. If the direction be defined by the cosines (l, m, n), and k be the coefficient of aberration, the centre of the sphere is virtually moved to the point $(-kl, -km, -kn)$. Thus the coordinates of a point (x, y, z) become

$$p(x + kl), \ p(y + km), \ p(z + kn),$$

p being a constant introduced in order that the sum of the squares may remain unity; *i.e.*

$$p^2 + 2p^2 k(ln + my + nz) + p^2 k^2 = 1$$

Now $k = 20''{\cdot}5$ and $k^2 = 0''{\cdot}002$, which is insensible in practice.

Thus neglecting k^2, we have

$$p = 1 - k \cos \theta$$

where $\cos \theta = lx + my + nz$, so that θ is the "Earth's Way," the angle between the direction of the star and that of the Earth's motion; and to the first order of k we have for the increments of x, y, z due to aberration

$$k(l - x \cos \theta), \ k(m - y \cos \theta), \ k(n - z \cos \theta).$$

Thus the increment of any coordinate x due to precession, nutation, and aberration, is

$$kl + a_1 x + a_2 y + a_3 z - kx(lx + my + nz),$$

and as regards the calculation of star-corrections, we do not appear to gain anything by the use of direction cosines.

18. It may be remarked incidentally that the effect of aberration on "standard coordinates"

$$\xi = \frac{x}{z} \text{ and } \eta = \frac{y}{z}$$

is to change them into

$$\xi^1 = \frac{x + kl}{z + kn} \text{ and } \eta^1 = \frac{y + km}{z + kn},$$

the common factor p disappearing in the division and leaving only the constants (for all stars) kl, km, kn. When we are dealing with a small photographic plate for which z is nearly unity, we may put $z+kn=z(1+kn)$, and $kl/(z+kn)=kl$,
and thus $\xi'=\xi(1-kn)+kl$.

There is thus a displacement of the centre by the quantities (kl, km) and a change of scale, ξ and η being both multiplied by $(1-kn)$. This result is derived in another way in *Monthly Notices*, vol. liv. p. 20.

Summary.

The present paper points out the advantages of using direction cosines or rectangular coordinates in certain astronomical computations, instead of curvilinear coordinates.

(*a*). In mapping the surface of the Sun or Moon, observations can be easily corrected so as to give us an orthogonal projection of the surface on a diametral plane : which is the same as giving two of three direction cosines. The third can be easily found, and then the transformation to any other axis is made by linear formulæ.

(*b*). In applying precession for long periods to star places, the formulæ in terms of direction cosines are very simple, and a small amount of tabulation would render the accurate reduction of one catalogue to another a simple matter.

The Extra-Equatorial Currents of Jupiter in 1899. By Rev. T. E. R. Phillips.

Despite the southern declination of *Jupiter* in 1899, an immense amount of detail was visible on the planet's surface, and numerous observations have been received from several observers, which have enabled the rotation periods of the surface material in various latitudes to be determined with, it is believed, considerable accuracy. The writer secured a large number of transits of spots during the earlier and intermediate months of the apparition. Mr. Denning, most fortunately, was able to pursue his observations very late—in some cases up to the middle of September; Mr. A. S. Williams has forwarded a large and valuable series of transits; and, in addition to these, much assistance has been derived from the figures of Messrs. Gledhill, Antoniadi, and J. Comas Solà, of Català, Spain (the observations of the latter being published in *Astronomische Nachrichten*, No. 3596, Band 150). Many of the spots have thus been followed for considerable periods of time (in some cases for more than seven months), and in the majority of cases the observations are so numerous and so accordant as to make the question of identi-